Quantum Near-Extremal Black Holes

Abstract

We review aspects of the Euclidean gravitational path integral in the context of the Reissner-Nordström black hole. We show how the tree-level path integral can be used to derive central results of black hole thermodynamics, following the method of Gibbons and Hawking. We then turn to the one-loop correction to the path integral. In the zero-temperature limit, the near-horizon geometry is given by $\text{AdS}_2 \times S^2$, and the path integral admits an infinite number of zero modes. We discuss how this leads to a $\log T$ contribution to the logarithm of the partition function at low temperature, which implies that the path integral is dominated by quantum corrections in the low-temperature regime.

Acknowledgements

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Contents

Notation and Conventions

In this essay, we work with 'mostly-plus' metric signature $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$. We use Planck units with $c = \hbar = G = k_B = 1$. The black hole manifold is given an orientation such that $\epsilon_{tr\theta\phi} = +\sqrt{|g|}$.

1 Introduction

Quantum field theory and general relativity are among the most successful scientific theories developed by humankind. While quantum field theory is the basis of particle physics and the Standard Model, general relativity is extraordinarily successful at describing gravitational effects, from the physics of our solar system all the way up to cosmological scales.

Despite their individual success, general relativity and quantum theory are famously at odds with each other. To date we do not have a universally accepted quantum theory that describes gravity at all energy scales. However, this does not mean that we have no idea how quantum field theory and general relativity interact. In the low energy regime, we can treat the metric as a quantum field and analyse it within the framework of effective field theory. Not only does this help us gain conceptual insights into quantum gravity; it also allows us to obtain physically relevant low-energy results that are agnostic to the full theory of quantum gravity.

These effects are especially important in situations where gravitational effects are strong, with black holes being notable examples. The analysis of quantum effects in the presence of black holes has led to many advances in theoretical physics. For example, it was discovered that black holes are thermodynamic objects that radiate. As such, they are associated with a temperature [1, 2] and an entropy [3].

The (Euclidean) *gravitational path integral* is an especially important tool in this context. Not only does the path integral formulation make perturbative expansion in the sense of effective field theory easily accessible, it also circumvents many of the problems that canonical quantisation suffers from [4]. Gibbons and Hawking applied the path integral to black holes in their seminal paper [5], providing an elegant derivation of the black hole entropy and other quantities of physical interest. More progress has been made in the following decades, including computations of one-loop terms in the gravitational path integral. These calculations continue to yield surprising physical results. For example, more recent developments have shown that the behaviour of low-temperature black holes can unexpectedly be *dominated by quantum effects* (see [6, 7, 8]).

In this essay, we review central aspects of the aforementioned results by investigating the gravitational path integral through the example of the Reissner-Nordström black hole. We start by setting the stage and establishing important classical properties of black holes in section 2. We discuss the extremal limit (which we will later identify as the zero-temperature limit) and find that it is characterised by a unique symmetry enhancement in the near-horizon region.

In section 3, we analyse the results obtained by Gibbons and Hawking [5]. We discuss how we can assign a temperature and entropy to the black hole, and how this relates to the tree-level contribution to the Euclidean gravitational path integral.

We will see that these results disagree with our physical intuition in the low-temperature regime. To resolve this, we turn to the one-loop quantum corrections to the path integral in sections 4 and 5, following mainly [7, 8, 6]. We find that, in the case of the extremal black hole, the enhanced symmetry near the horizon gives rise to a special class of fluctuations in the path integral called *zero modes*. They lead to the near-extremal black hole being unexpectedly dominated by quantum corrections over the familiar classical result.

2 The Charged Black Hole

In this section, we review properties of (Lorentzian) black holes in classical general relativity relevant to this essay, following mainly [9]. We introduce the Reissner-Nordström solution describing a charged black hole and discuss its extremal limit, which will be a central object of interest in the later sections about the gravitational path integral. To close the section, we briefly discuss the laws of black hole mechanics. We will encounter them again when discussing quantum properties of the Euclidean black hole.

2.1 Reissner-Nordström solution

Despite their importance to theoretical physics, black hole solutions in general relativity are surprisingly constrained: the celebrated *No-hair theorem* states that any stationary, axisymmetric and asymptotically flat black hole solution to the vacuum Einstein equations is fully characterised by just two parameters: the mass M and angular momentum J . If an electromagnetic field is included, one obtains two additional parameters Q and P that describe the electric and magnetic charge of the black hole, respectively [9].

In this essay, we are mainly interested in spherically symmetric black holes in $d = 4$ spacetime dimensions. Spherical symmetry implies that the angular momentum parameter J must necessarily vanish, since the axis of rotation would otherwise single out a preferred spatial direction[1]. The resulting solution is called the Reissner-Nordström (RN) black hole, characterised by the three parameters (M, Q, P). In this section, we present this solution and some of its most important properties, following the discussion in [9, 10].

The action (in Lorentzian signature) is given by the Einstein-Maxwell action

$$S_{EM} = \frac{1}{16\pi} \int \mathrm{d}^4x\sqrt{-g}\left(R - F^{ab}F_{ab}\right) , \tag{2.1}$$

where $F = \mathrm{d}A$ denotes the electromagnetic field strength. The equations of motion are

$$\begin{aligned} &R_{ab} - \frac{1}{2}g_{ab}R = 2\left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right) , \\ &\mathrm{d}{\star}F = 0 \qquad \mathrm{d}F = 0 \, . \end{aligned} \tag{2.2}$$

The absence of electric and magnetic currents in the Maxwell equations reflects that the Reissner-Nordström solution has vanishing electromagnetic currents outside of the singularity. The equations of motion are solved by

$$\begin{aligned} &ds^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 , \\ &f(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2} , \quad e^2 = Q^2 + P^2 , \\ &A = -\frac{Q}{r}\mathrm{d}t - P\cos\theta\,\mathrm{d}\phi , \end{aligned} \tag{2.3}$$

where $\mathrm{d}\Omega^2$ denotes the metric on the 2-sphere and we take $P, Q \geq 0$ without loss of generality. Note that this solution reduces to the well-known Schwarzschild metric in the case $Q = P = 0$.

[1] Note that this argument does not apply in two spatial dimensions. Since the rotation group $SO(2)$ is only one-dimensional, a non-zero J does not specify a plane or direction of rotation. This rotating solution in $d = 1 + 2$ is called the BTZ black hole, named after Bañados, Teitelboim and Zanelli.

As advertised above, the parameter M describes the mass of the black hole[2], while Q and P are its electric and magnetic charge. The first statement follows by matching the metric (2.3) to the metric in the Newtonian approximation of general relativity as $r \to \infty$. To see that Q and P describe the electric and magnetic charge, recall that the general form of the Maxwell equations reads

$$\mathrm{d}{\star F} = 4\pi {\star J_e} \ , \quad \mathrm{d}F = 4\pi {\star J_m} \ , \tag{2.4}$$

if the electric and magnetic currents J_e, J_m are suitably normalised. We can then define the total electric and magnetic charge by integrating the 3-forms $\star J_{e,m}$ over a spatial slice Σ of constant t

$$Q_{BH} = \int_\Sigma \star J_e \ , \quad P_{BH} = \int_\Sigma \star J_m \ . \tag{2.5}$$

Using Stokes' theorem, we can rewrite

$$Q_{BH} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2(r)} \star F \ , \quad P_{BH} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2(r)} F \ . \tag{2.6}$$

This can be evaluated directly using (2.3) and

$$F = \mathrm{d}A = \frac{Q}{r^2} \mathrm{d}r \wedge \mathrm{d}t + P \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{2.7}$$

to yield $Q_{BH} = Q$ and $P_{BH} = P$, confirming that Q and P do indeed describe the electric and magnetic charges of the spacetime.

By computing curvature scalars, one finds that there is a singularity at the origin. To find the location of the horizons, we analyse the sign of the $f(r)$ function in (2.3). Writing

$$r^2 f(r) = (r - r_+)(r - r_-) \ , \quad r_\pm = M \pm \sqrt{M^2 - e^2} \ , \tag{2.8}$$

we can see that the qualitative behaviour of $f(r)$ depends on the relative magnitude of e and M. If $M > e$, there are two horizons $r_\pm$ with the Killing field $(\partial/\partial t)^a$ becoming spacelike in the region $r_- < r < r_+$. Any observer that crosses the outer horizon is drawn into the black hole and necessarily reach the inner horizon within finite proper time [10].

Conversely, if $M < e$, the function $f(r)$ is positive definite and there is no horizon. This is an example of a *naked singularity*, and the cosmic censorship conjecture states that these objects do not form in our universe.

Throughout this essay, an important limit is the special case where the mass M and the 'electromagnetic charge' e are precisely equal. This is called the *extremal* black hole.

2.2 The extremal limit

The limit $M = e$ is called the *extremal* Reissner-Nordström solution. In this case, the locations of the two horizons coincide and $f(r)$ has a double zero,

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{M}{r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \ . \tag{2.9}$$

[2] We assume $M > 0$.

This metric has a few notable properties. Firstly, the Killing field $(\partial/\partial t)^a$ is timelike everywhere (except at the horizon): the usual 'sign-flip' in the metric components that draws infalling observers further to the centre of the black hole is absent. It is important to note, however, that an observer who fell into the black hole cannot cross the event horizon again in the opposite direction and return to their starting position. This becomes clear from the associated Penrose diagram [9].

Secondly, the extremal black hole exhibits a special geometry in the near-horizon region. We can define the extremal radius $r_0 \equiv Q = M$ and the new radial coordinate $\rho = r - r_0$. Rescaling time as $t \to r_0^2 t$ and expanding the metric in powers of ρ/r_0, we find that to leading order [7]

$$ds^2 = r_0^2 \left(-\rho^2 \mathrm{d}t^2 + \frac{1}{\rho^2}\mathrm{d}\rho^2 + \mathrm{d}\Omega^2 \right) . \tag{2.10}$$

We can identify this near-horizon geometry as the metric of two-dimensional Anti de Sitter space, with a 2-sphere of radius r_0 attached to every point. AdS_2 is what is known as a maximally symmetric space, i.e. it has three *Killing vectors*, the maximum number for a two-dimensional space. We can see this by solving the Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{2.11}$$

for the AdS$_2$ part. Using the explicit form of the metric (2.10), we indeed find three solutions

$$\begin{aligned}
H^a &= \partial_t && \text{(time translation)},\\
D^a &= t\,\partial_t - \rho\,\partial_\rho && \text{(dilation)},\\
K^a &= \left(t^2 + 1/\rho^2\right)\partial_t - 2t\rho\,\partial_\rho && \text{(special conformal)}.
\end{aligned} \tag{2.12}$$

They satisfy the commutation relations

$$[H, D] = H\,, \quad [H, K] = 2D\,, \quad [D, K] = K\,, \tag{2.13}$$

which are precisely the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Thus, the near-horizon geometry AdS$_2 \times S^2$ has the 6-dimensional isometry group[3] $SL(2, \mathbb{R}) \times SO(3)$. Comparing this with the 4-dimensional isometry group $\mathbb{R} \times SO(3)$ of the full Reissner-Nordström solution (with $\mathbb{R}$ denoting time translations), we find that there is a *symmetry enhancement* in the near-horizon region [7]. This will later play a prominent role in the path integral of the black hole.

Thirdly, consider a line of constant t, θ, ϕ connecting the horizon $r = r_0$ with a point $r = R > r_0$. The proper length of this line is given by

$$L = \int_{r_0}^{R} \mathrm{d}r \sqrt{g_{rr}} = \int_{r_0}^{R} \frac{\mathrm{d}r}{1 - r_0/r} = \infty\,. \tag{2.14}$$

We see that the horizon is located at an 'infinite distance' from a point in the outer-horizon region[4]; we say that the throat connecting the horizon with the asymptotically flat region $r \gg r_0$ becomes 'infinitely long' [9] (see Fig. 1). This is different from the non-extremal Reissner-Nordström solution, where the length L remains finite.

[3] Strictly speaking, this argument is incomplete since multiple (topologically different) Lie groups share the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra. In this essay, we do not distinguish them since global properties of the symmetry group are not relevant to our discussion.

[4] The *proper time* to reach the horizon for an infalling observer is still finite, however.

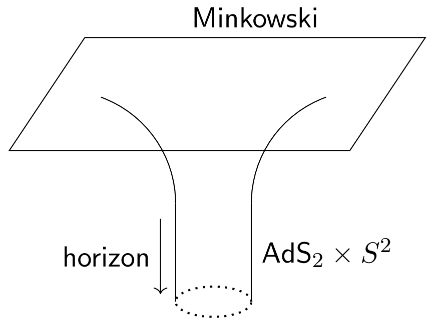

Figure 1: The AdS$_2 \times S^2$ throat becomes infinitely long in the extremal limit.

2.3 Black hole mechanics

To close this section on classical (Lorentzian) black holes, we briefly state the laws of black hole mechanics [11], following the discussion in [12, 9]. These are rules derived in classical general relativity that bear a close resemblance to the laws of thermodynamics.

To describe the properties of the black hole horizon, we define the *surface gravity* κ such that

$$\kappa\xi^a = \xi^b \nabla_b \xi^a \quad \text{at the horizon.} \tag{2.15}$$

For the RN black hole, ξ^a is the Killing field corresponding to time translation normalised such that $\xi^a\xi_a = -1$ at infinity. This results in a surface gravity of

$$\kappa_{RN} = \frac{f'(r_+)}{2} \; . \tag{2.16}$$

κ is called surface gravity because it describes the force per unit mass an observer at infinity would need to exert on a mass to hold it at rest at the horizon [9].

Using these definitions, one can show that the following laws of black hole mechanics hold in classical general relativity. More details can be found in [9].

- (Zeroth Law) The surface gravity κ is constant over the horizon of the RN black hole[5].
- (First Law) If the mass, charge and horizon area of the RN black hole are linearly perturbed, we have

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Phi_H \mathrm{d}Q \; , \tag{2.17}$$

  where Φ_H denotes the difference in the electric potential between the horizon and infinity.

- (Second Law) The horizon area of the RN black hole $A = 4\pi r_+^2$ can only increase with time,

$$\frac{\mathrm{d}}{\mathrm{d}t} A \geq 0 \; . \tag{2.18}$$

These laws are clearly reminiscent of the laws of thermodynamics, with the surface gravity playing the role of temperature and the area playing the role of entropy. At the moment, this interpretation may seem quite

[5]Of course, this result follows immediately from the spherical symmetry of the Reissner-Nordström solution. However, the zeroth law also holds for other black holes that are not spherically symmetric.

mysterious. Classically, black holes are cold objects with zero temperature, so it appears bizarre to give them a thermodynamic interpretation. However, starting in the next section, we will see what happens when quantum theory enters the picture, and how we can use the (Euclidean) gravitational path integral to make the analogy to thermodynamics much more precise.

For completeness, let us mention that there is also a notion of a third law of black hole mechanics, though its validity and exact formulation are highly debated [13, 14].

3 The Euclidean Path Integral at Tree Level

Classical general relativity might make us think that we understand black holes quite well. After all, they are heavily constrained by symmetries and many calculations are analytically tractable. But the success of quantum field theory enabled physicists to uncover surprising new properties of black holes that challenge many of our most fundamental beliefs about the nature of these objects. Hawking discovered that, in the presence of quantum fields, black holes emit thermal radiation and are thus associated with a *Hawking temperature* [1]. Thus, paradoxically, black holes are not actually black. Furthermore, Bekenstein argued that one can assign black holes an *entropy* [3]. As we would expect from the laws of black hole mechanics (section 2.3), the Hawking temperature is proportional to the surface gravity, while the Bekenstein-Hawking entropy is proportional to the black hole horizon area. These results sharpen the analogy between black holes and thermodynamic systems, turning the laws of black hole mechanics into the laws of *black hole thermodynamics* [12].

Gibbons and Hawking realised the power of the Euclidean path integral in describing black holes [5]. In this section, we use their approach to derive the Reissner-Nordström black hole entropy and temperature. After a brief introduction to Wick rotation and Euclidean field theory, we show how the black hole temperature follows directly from regularity conditions on the path integral. Subsequently, we derive the entropy by calculating the tree-level partition function. Strikingly, the predictions from the Euclidean theory agree with the expectations from the Lorentzian theory and the laws of black hole mechanics.

3.1 Wick rotation and Euclidean quantum field theory

The method pioneered by Gibbons and Hawking relies on Euclidean metrics, which arise from their Lorentzian counterparts by a Wick rotation $t = -i\tau$. There are multiple good reasons to consider Euclidean black holes. Firstly, Wick rotation is a well-known trick in quantum field theories to improve the convergence properties of integrals. To see this, consider a quantum mechanical particle moving in one dimension. The path integral is given by

$$\langle x_f| \, e^{-iHt} \, |x_i\rangle \propto \int_{x_i}^{x_f} \mathcal{D}x \exp\left(iS\right) \; , \quad S = \int \mathrm{d}^4x \left(\frac{m}{2}\dot{x}^2 - V(x)\right) \; . \tag{3.1}$$

After Wick rotation, this expression becomes

$$\langle x_f| \, e^{-H\tau} \, |x_i\rangle \propto \int_{x_i}^{x_f} \mathcal{D}x \exp\left(-S_E\right) \; , \quad S_E = \int \mathrm{d}^4x_E \left(\frac{m}{2}x'^2 + V(x)\right) \; , \tag{3.2}$$

with a prime denoting a derivative with respect to Euclidean time τ. We thus traded the oscillatory integral for an exponentially decaying integral over a (positive definite) Euclidean action S_E, which is much easier to evaluate. This approach - considering the Euclidean theory instead of the Lorentzian and only taking the inverse Wick rotation at the very end of the calculation - has proved successful in many quantum field theories [15]. It is reasonable to expect that this will also simplify the gravitational path integral.

A second reason to study the Euclidean theory is a surprising connection between Euclidean quantum field theory and statistical physics [15]. Upon inspection of (3.2), one notices that the left hand side has the form of a *partition function*

$$Z = \mathrm{Tr}\, e^{-\beta H} \tag{3.3}$$

if we identify the inverse temperature with Euclidean time $\beta = \tau$ and sum over a complete basis of eigenstates,

$$Z = \mathrm{Tr}\, e^{-H\tau} \propto \int^{x(\tau)=x(0)} \mathcal{D}x \exp\left(-S_E\right) \ . \tag{3.4}$$

We thus find that a field theory at finite temperature can be viewed as a Euclidean field theory periodic in time, with period $\tau = \beta$. This is directly relevant to our case of interest, since we already established that quantum black holes are thermodynamic objects with finite temperature.

There is, however, some cause for concern when it comes to applying Wick rotations to a field theory on a curved background. After all, performing a Wick rotation requires selecting a preferred time coordinate, which seems to contradict the principles of general relativity. While this does indeed pose difficulties for general spacetimes, our black hole solution has Killing field $(\partial/\partial t)^a$ that is timelike (outside the horizon), which intuitively singles out a 'preferred' time direction [16]. Furthermore, Wick rotation does not guarantee that the Euclidean action S_E is positive definite. Indeed, the *conformal mode problem* states that the Euclidean action is unbounded from below in many gravitational theories of interest, leading to a divergent integral [17]. We will discuss this issue again in section 4.3.

3.2 The Hawking temperature

Let us apply this idea to our favourite example, the Reissner-Nordström black hole. Under a Wick rotation, the solution becomes (denoting Euclidean time with t) [6]

$$\begin{aligned} &ds^2 = f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \ , \\ &f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \ , \quad A = iQ\left(\frac{1}{r} - \frac{1}{r_+}\right)\mathrm{d}t \ , \quad F_{tr} = -F_{rt} = i\frac{Q}{r^2} \ . \end{aligned} \tag{3.5}$$

Here, we set the magnetic charge $P = 0$ for simplicity. Furthermore, we performed a *gauge transformation* on the field A that shifts the t-component by a constant. Note that this transformation is consistent with the $\mathbb{R} \times SO(3)$ symmetry of the system. The reason for this gauge transformation will become clear shortly.

According to the discussion in section 3.1, we should think of the time coordinate t as periodic with periodicity $\beta = 1/T$. This could potentially be problematic and lead to singularities in the metric (3.5) at the zeros of the function $f(r)$, i.e. at the black hole horizon. To see this in detail, focus on the region just outside the horizon r_+ and introduce the coordinate $\delta r = r - r_+$. Ignoring the angular part and discarding terms of order $\mathcal{O}(\delta r/r_+)^2$, we have [12, 14]

$$\begin{aligned} ds^2 &= \frac{\delta r}{r_+}\left(1 - \frac{r_-}{r_+}\right)\mathrm{d}t^2 + \frac{r_+}{\delta r}\left(1 - \frac{r_-}{r_+}\right)^{-1}(\mathrm{d}\delta r)^2 \\ &= \delta r f'(r_+)\mathrm{d}t^2 + (\delta r f'(r_+))^{-1}(\mathrm{d}\delta r)^2 \ . \end{aligned} \tag{3.6}$$

Substituting $\delta r = \rho^2$, we find that

$$ds^2 = \text{const} \cdot \left(\frac{\rho^2}{4}f'(r_+)^2\mathrm{d}t^2 + \mathrm{d}\rho^2\right) \ , \tag{3.7}$$

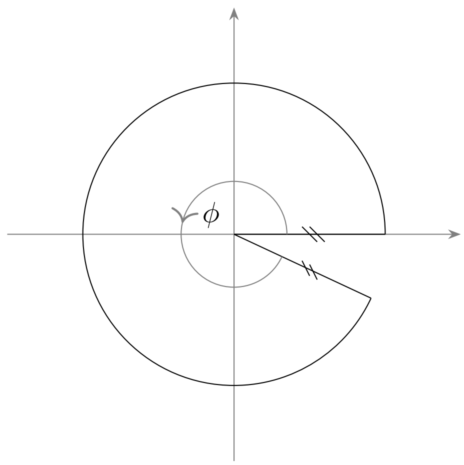

Figure 2: Illustration of the conical singularity. In polar coordinates, a periodicity different from 2π (a deficit angle) leads to a singularity at the origin, corresponding to the tip of the cone.

so the metric in the near-horizon region is proportional to the line element of ordinary two-dimensional Euclidean space in polar coordinates, with ρ denoting a radial coordinate and t an angle.

In order to avoid a singularity at the origin, the periodicity β has to be chosen exactly as in polar coordinates. Any other choice leads to a so-called *conical singularity* [12], as illustrated in Fig. 2. This means that the angle $\phi = f'(r_+) \cdot t/2$ must have a period of 2π, or

$$t \sim t + \frac{4\pi}{f'(r_+)} = t + \beta \, . \tag{3.8}$$

Therefore, to avoid singularities in the Euclidean black hole metric, the *Hawking temperature* of the black hole must be

$$T = \frac{f'(r_+)}{4\pi} = \frac{r_+ - r_-}{4\pi r_+^2} \, . \tag{3.9}$$

Comparing with (2.16), we see that the temperature is proportional to the surface gravity

$$T = \frac{\kappa}{2\pi} = \frac{\hbar\kappa}{2\pi c k_B} \, , \tag{3.10}$$

as advertised above. The explicit appearance of $\hbar$ makes it clear that this radiation is a quantum effect. Even though the derivation of the black hole temperature given here may seem quite abstract and not fully convincing, it agrees with Hawking's result for the temperature of the particles emitted by black holes obtained using methods of quantum field theory in curved spacetime [1].

This result is remarkable. We discovered that the Euclidean black hole is smooth with topology $\mathbb{R}^2 \times S^2$, with Euclidean time representing the angular coordinate in $\mathbb{R}^2$. This is called the 'cigar' geometry (see Fig. 3), and the correct black hole temperature ensures that the cigar is smooth at its tip [14]. Note that we seemingly eliminated the region inside the horizon ($r < r_+$). This region contains the curvature singularity and the Euclidean metric is not positive definite here, so we drop it and describe the black hole purely by its geometry outside the horizon [5].

We must also be careful to ensure that the additional fields we introduce are regular at $r = r_+$. From (3.5) we see that the gauge field A points in the angular t-direction. To ensure that it is continuous, its

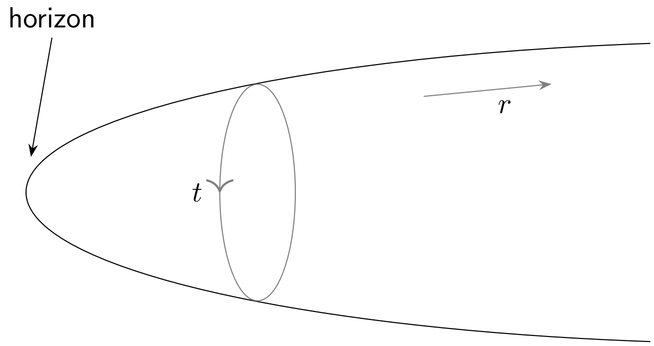

Figure 3: The Euclidean cigar black hole. The correct choice of temperature T, which fixes the angular periodicity, ensures that the geometry is smooth at the tip of the cigar.

magnitude has to vanish at $r = r_+$ [18]. This explains the gauge transformation performed in (3.5).

We can now understand the relevance of the extremal limit discussed in section 2.2. The temperature (3.9) is lowered if the horizons r_+, r_- move closer together, and the black hole reaches zero temperature precisely in the *extremal limit*. In this state, the black hole cannot emit radiation; in this sense, the extremal limit is the (classical) *ground state* of the Reissner-Nordström black hole [19].

3.3 The Euclidean action and the Bekenstein-Hawking entropy

Having understood the metric and matter configuration that constitute the Euclidean RN solution, we can now turn to the gravitational path integral. In this section, we follow Gibbons and Hawking [5]. The central object of interest is the partition function

$$Z = \mathcal{N} \int \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu \, \exp(-S_E[g, A]) \, . \tag{3.11}$$

The full path integral is hard to make sense of, but we can hope to make progress with an approximation using the solution to the classical equations of motion. This is because the classical solution minimises the Euclidean action, making it the dominating contribution to (3.11). To leading order, we simply have

$$Z = \exp(-S_E[\bar{g}, \bar{A}]) \, , \tag{3.12}$$

where $\bar{g}$ and $\bar{A}$ denote the solutions to the classical equations of motion (3.5). Thus, our task is to evaluate this classical action.

We study the black hole as a thermodynamic system in the *canonical ensemble*. That means that we hold the total charge Q and temperature T fixed [8], e.g. when varying the action to obtain the equations of motion.

How do we implement this choice of ensemble in the action? The idea is to think of the Euclidean cigar as a manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$ at some large radius $r = R_b \gg r_+$ (which we will eventually take to infinity) and to include appropriate boundary terms in the action. The purpose of these terms is to ensure that the correct equations of motions are reproduced if we vary the action in a way that respects the canonical ensemble. To achieve this, they must cancel the boundary integrals that arise from integration by parts. The full action (in Euclidean signature) is given by

$$\begin{aligned} S_E &= S_{EM} + S_{boundary} \\ &= -\frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}\left(R - F^{ab}F_{ab}\right) - \frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{h}\left(K + 2n_aA_bF^{ab}\right) \ . \end{aligned} \tag{3.13}$$

Here, n_a denotes the outward normal to the boundary, K the extrinsic curvature of the boundary and h the determinant of the induced metric. The term including the extrinsic curvature is called the Gibbons-Hawking-York (GHY) boundary term, and it ensures that varying with respect to the metric leads to the Einstein equations as long as $\delta g\big|_{\partial\mathcal{M}} = 0$ [5]. The term including the gauge field ensures that variation with respect to A reproduces the Maxwell equations, provided the charge Q (or equivalently the flux $F^{\mu\nu}$ at the boundary) is fixed [20, 6]. Thus, we are imposing *Dirichlet* boundary conditions for the metric and *Neumann* boundary conditions for the gauge field.

This expression $S_E[\bar{g}]$ for the on-shell action can be evaluated explicitly using the classical solution (3.5) [21]. The metric induced on the boundary and the corresponding normal are given as

$$h = f(R_b)\mathrm{d}t^2 + R_b^2\mathrm{d}\Omega^2 \ , \quad n = \sqrt{f(R_b)}\,\partial_r \tag{3.14}$$

in terms of the volume element of the 2-sphere $\mathrm{d}\Omega^2$. The extrinsic curvature can be evaluated using the standard formula [21]

$$\sqrt{h}K = n^\mu\partial_\mu\sqrt{h} \ . \tag{3.15}$$

Furthermore, note that the Einstein-Hilbert term simply vanishes. The trace of the Einstein equations tells us that the Ricci scalar is proportional to the trace of the energy-momentum tensor. The latter vanishes in four dimensions for the electromagnetic field. Thus, $R = 0$ (cf. [22]).

Taking equations (3.5), (3.14) and (3.15) together, the result for the action is

$$\begin{aligned} S_E[\bar{g}] &= \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{g}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4\pi}\int \mathrm{d}^3y\sqrt{h}n_\mu A_\nu F^{\mu\nu} - \frac{1}{8\pi}\int \mathrm{d}^3y\sqrt{h}K \\ &= \frac{\beta Q^2}{2r_+} - \frac{\beta}{2}\left(2R_b - 3M + \frac{Q^2}{R_b}\right) \ , \end{aligned} \tag{3.16}$$

with the first part coming from the terms involving A_μ, and the second part coming from the GHY boundary term.

Note that this action is divergent in the limit $R_b \to \infty$ due to the infinite volume of space. Thus, it needs to be regularised appropriately. A convenient way to achieve this is to define the action relative to a 'reference background' by subtracting off $S_E[g_0]$, the action for standard Euclidean space $g_0 = \mathrm{d}t_0^2 + \mathrm{d}r_0^2 + r_0^2\mathrm{d}\Omega^2$ with periodic time but without a black hole [5, 21, 12]. Since there is no gauge field present, this action is just given by the boundary term

$$S_E[g_0] = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{h_0}K_0 \ , \tag{3.17}$$

with h_0 and K_0 denoting the induced metric and extrinsic curvature of the boundary $\partial\mathcal{M}$ in the Euclidean reference space. In order to define the regularisation consistently, the (proper) length of the periodic time direction must be equal for the black hole and reference spacetime [21]. The relevant condition is

$$\int_0^\beta \mathrm{d}t\sqrt{g_{tt}} = \int_0^{\beta_0} \mathrm{d}t_0\sqrt{(g_0)_{tt}} \quad \text{at } r = R_b \ , \tag{3.18}$$

which implies that the time periodicity β_0 in the reference spacetime must be chosen such that

$$\beta_0 = \beta\sqrt{f(R_b)} = \beta\left(1 - \frac{2M}{R_b} + \frac{Q^2}{R_b^2}\right) . \tag{3.19}$$

The action can now be evaluated as above, leading to the result

$$S_E[g_0] = -R_b\beta_0 = -R_b\beta\sqrt{1 - \frac{2M}{R_b} + \frac{Q^2}{R_b^2}} . \tag{3.20}$$

The divergent piece in (3.16) precisely cancels when we subtract off the action of this reference background, and we can safely take R_b to infinity. The final, regularised action is

$$\log Z = -S_E^{reg} = -S_E[\bar{g}] + S_E[g_0] = -\beta\left(\frac{M}{2} + \frac{Q^2}{2r_+}\right) . \tag{3.21}$$

Since we are in the canonical ensemble with fixed (Q, T), both M and r_+ should be thought of as functions of charge and temperature.

We can calculate the energy and entropy of this spacetime by using the usual thermodynamic relations [21] for the canonical ensemble and substituting r_+ and M as a function of β and Q (cf. (3.9), (2.8)). We find

$$E \equiv -\frac{\partial}{\partial\beta}\log Z = M , \quad S \equiv \log Z + \beta E = \pi r_+^2 . \tag{3.22}$$

We recognise the *area* $A = 4\pi r_+^2$ of the outer horizon of the black hole in the second equation. Thus, the black hole entropy as derived from the tree-level Euclidean path integral is

$$\boxed{S_{BH} = \frac{A}{4} = \frac{k_B c^3 A}{4G\hbar} ,} \tag{3.23}$$

which is the famous result due to Bekenstein and Hawking [3]. The independent confirmation of this formula is a major success of the gravitational path integral approach [5].

These results have a particularly natural interpretation in the context of the laws of black hole mechanics [12] (section 2.3). With the input of the gravitational path integral, the zeroth law turns into the statement that the black hole temperature is constant over the horizon, while the second law says that the entropy cannot decrease with time. Furthermore, we can substitute the results (3.10) and (3.22) into the first law (2.17) to find

$$\mathrm{d}E = T\mathrm{d}S + \Phi_H\mathrm{d}Q , \tag{3.24}$$

which is nothing but the familiar first law of thermodynamics. Thus, the inclusion of quantum effects in the analysis of black holes improves the analogy between black holes and thermodynamic systems [14]. The Euclidean path integral is an important tool in this context, and many results originally obtained from the Lorentzian black hole can be derived from the Euclidean theory.

Let us analyse the entropy formula (3.23) a bit closer. The fact that S_{BH} depends only on the horizon area is remarkable; in some vague sense, the entropy seems to be a quantity associated purely to the near-horizon region [6]. This is reflected in our derivation by imposing regularity of fields at the horizon, which crucially fixes the values of both the temperature T and the gauge field A_μ in (3.5) (and by extension, the entropy). We will briefly comment on this notion of entropy as a near-horizon quantity again at a later point.

As nice as this result may be, it raises some questions about the nature of the extremal black hole [19, 6, 23]. Recall that the entropy in statistical mechanics can be interpreted as a count of microstates. The entropy in the zero-temperature limit thus measures the *ground state degeneracy* of the system. We would typically expect this degeneracy to be small. This is the underlying principle behind the third law of thermodynamics, which (in one of its many variants) states that the entropy vanishes in the zero-temperature limit.

This is radically different from our prediction (3.23). In the zero-temperature limit, the black hole approaches the extremal solution, which has far from low entropy. To give a rough estimate, an extremal black hole with the mass of our sun (and equally large charge) would have an entropy of

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar} \sim 10^{76} \,, \tag{3.25}$$

which makes a mockery of the aforementioned naive notion of the third law of thermodynamics.

There are at least two ways out of this dilemma. Firstly, it is conceivable that black holes really do have a highly degenerate ground state, and perhaps this is explained by some yet unknown physical mechanism [23]. Another possibility is that the entropy formula (3.23) is untrustworthy in the first place, at least in the low-temperature regime. Remember that (3.23) was derived using a zeroth-order approximation to the path integral (3.11) and we expect there to be higher-order corrections. To understand these corrections at zero temperature, we must tackle the path integral for the extremal black hole at the next order in perturbation theory.

Evaluating these corrections will be the central task for the remainder of this essay. This will require us to think more deeply about the structure of the gravitational path integral to understand the true nature of the near-extremal black hole.

4 One-Loop Path Integral of the Extremal Black Hole

To understand the unusual behaviour of the black hole at low temperature, we turn to the one-loop contribution to the extremal black hole partition function, following mainly [7, 6, 8]. This correction involves integration over quantum fluctuations of the metric and the gauge field. We start by making this integration explicit and discussing general properties of the path integral. Following that, we discover that the path integral of the extremal black hole contains an infinite number of special fluctuations called *zero modes*, and we discuss the implications of this result.

4.1 Saddle point approximation

Let us first discuss how to evaluate the path integral in perturbation theory. Recall that in section 3.3 we approximated the full partition function

$$Z = \mathcal{N} \int \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu \, \exp(-S_E[g, A]) \tag{4.1}$$

by identifying the extremal point of the action ($\delta S = 0$) as the leading contribution. To go one order higher in perturbation theory, we have to consider the fluctuations in the vicinity of the classical solution. This means we should *Taylor-expand* the action about the classical solution as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ , \quad A_\mu = \bar{A}_\mu + \frac{1}{2} a_\mu \ . \tag{4.2}$$

Schematically, we have

$$S_E[g, A] \approx S_E[\bar{g}, \bar{A}] + S_E^{(1)}\Phi + \frac{1}{2}\Phi S_E^{(2)}\Phi + \mathcal{O}(\Phi^3) \ , \quad \Phi \in \{h, a\} \ . \tag{4.3}$$

The constant term factors out of the path integral, yielding the same contribution already calculated in section 3. The linear term vanishes by construction, since we are expanding about a stationary point. The leading quantum correction is hence given by the term quadratic in the fields. The path integral thus reduces to a Gaussian integral, and the solution is given by the determinant of the differential operator $S_E^{(2)}$. This is called the *one-loop determinant*. Schematically, we have [6]

$$Z_{\text{1-loop}} = \mathcal{N} \int \mathcal{D}\Phi \, \exp\left(-S_E[\bar{g}, \bar{A}] - \frac{1}{2}\Phi S_E^{(2)}\Phi \right) \sim \frac{\exp(-S_E[\bar{g}, \bar{A}])}{\sqrt{\det S_E^{(2)}}} \ . \tag{4.4}$$

This method of approximating the partition function is, for obvious reasons, called the *saddle point approximation*.

The first thing to do is to define an appropriate background $\bar{g}$, $\bar{A}$ for our case of interest. Recall that the extremal black hole has a special AdS$_2 \times S^2$ geometry (2.10) in the near-horizon region, and that this 'throat' is infinitely long in the extremal limit (see section 2.2). Following the treatment in [6, 7, 24], we define this AdS$_2 \times S^2$ metric as our background field $\bar{g}$. This metric splits into two relatively simple two-dimensional spaces, which makes calculations involving the complicated differential operator in (4.3) much more tractable.

One might doubt whether the AdS$_2 \times S^2$ geometry still gives physically reasonable results for our black hole, given that the metric (2.10) is only valid in the near-horizon region. Since we are considering the extremal case, our approach is justified because the AdS$_2 \times S^2$ throat is infinitely long and thus a good description of the entire black hole system[6]. This agrees with our intuition from the discussion around (3.23). We think of entropy as a quantity associated with the horizon region, so it seems plausible that focussing only on this region will still capture the relevant physics [6, 24].

4.2 Gauge fixing and ghosts

In order to make the path integral well-defined, there are several conceptual and technical issues to address. Note that we are dealing with a path integral involving *gauge symmetry*. On the one hand, the photon field A is a $U(1)$ gauge field. On the other hand, the action is invariant under diffeomorphisms, the gauge symmetry of general relativity. The path integral (4.1) integrates over all configurations (including gauge-equivalent configurations), leading to an over-counting. Another way to say this is that there is a large (infinite) number of flat directions in field space, i.e. modes for which the change in the action and hence the second order contribution in (4.3) vanishes. The familiar method to deal with this over-counting is due to Faddeev and Popov [25] and involves introducing additional gauge fixing and ghost terms into the Lagrangian.

[6] This is more subtle in the near-extremal case, where the throat is long but not infinitely long. This is discussed in section 5.

First, we have to decide on a gauge for the metric and gauge field fluctuations. The standard choices are [7]

$$\begin{aligned} &\text{Lorenz gauge:}\ \ \mathcal{F}(a) = \bar{\nabla}_\mu a^\mu = 0 \\ &\text{Harmonic gauge:}\ \ \mathcal{G}^\nu(h) = \bar{\nabla}_\mu h^{\mu\nu} - \frac{1}{2}\bar{\nabla}^\nu h = 0\ , \end{aligned} \tag{4.5}$$

where $\bar{\nabla}_\mu$ refers to the covariant derivative with respect to the background geometry. The Lorenz gauge condition is used in many quantum field theory contexts, and the harmonic gauge is familiar from the analysis of gravitational waves in linearised gravity.

The corresponding gauge fixing term in the action is [7, 6]

$$\mathcal{L}_{\text{gauge}} = \mathcal{L}_{\text{diff}} + \mathcal{L}_{U(1)} = \frac{1}{32\pi}\left(\bar{\nabla}_\mu h^{\mu\nu} - \frac{1}{2}\bar{\nabla}^\nu h\right)\left(\bar{\nabla}^\rho h_{\rho\nu} - \frac{1}{2}\bar{\nabla}_\nu h\right) + \frac{1}{32\pi}\left(\bar{\nabla}_\mu a^\mu\right)^2\ . \tag{4.6}$$

This contribution has to be added to the action (3.13). We evaluate the action expanded to second order using symbolic computation in Mathematica (xAct). We find (in agreement with [7, 6])

$$\begin{aligned} \mathcal{L}\ &= \mathcal{L}_h + \mathcal{L}_a + \mathcal{L}_{ha}\ ; \\ \mathcal{L}_h &= h^*_{\mu\nu} D^{\mu\nu\rho\sigma} h_{\rho\sigma} \\ &= -\frac{1}{16\pi} h^*_{\mu\nu}\left(\frac{1}{4}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\bar{\Box} - \frac{1}{8}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}\bar{\Box} + \frac{1}{2}\bar{R}^{\mu\rho\nu\sigma} + \frac{1}{2}\bar{R}^{\mu\rho}\bar{g}^{\nu\sigma} - \frac{1}{2}\bar{R}^{\mu\nu}\bar{g}^{\rho\sigma}\right. \\ &\quad \left. + \frac{1}{8}\bar{F}^2\left(2\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} - \bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}\right) - \bar{F}^{\mu\rho}\bar{F}^{\nu\sigma} - 2\bar{F}^{\mu\lambda}\bar{F}^{\rho}{}_{\lambda}\bar{g}^{\nu\sigma} + \bar{F}^{\mu\lambda}\bar{F}^{\nu}{}_{\lambda}\bar{g}^{\rho\sigma}\right) h_{\rho\sigma}\ , \\ \mathcal{L}_a &= a^*_\mu P^{\mu\nu} a_\nu = -\frac{1}{32\pi} a^*_\mu\left(\bar{g}^{\mu\nu}\bar{\Box} - \bar{R}^{\mu\nu}\right) a_\nu\ , \\ \mathcal{L}_{ha} &= h^*_{\mu\nu} T^{\mu\nu\rho\sigma}\bar{\nabla}_\rho a_\sigma + \text{h.c.} = \frac{1}{32\pi} h^*_{\mu\nu}\left(4\bar{g}^{\mu[\rho}\bar{F}^{\sigma]\nu} + \bar{F}^{\rho\sigma}\bar{g}^{\mu\nu}\right)\bar{\nabla}_\rho a_\sigma + \text{h.c.}\ , \end{aligned} \tag{4.7}$$

where barred quantities are evaluated with respect to the background metric.

The pure metric term $\mathcal{L}_h$ contains both contributions from the expansion of the Ricci scalar and diffeomorphism gauge fixing term (first line) as well as the variation of the F^2-term (second line). $\mathcal{L}_a$ comes from varying the F^2-term with respect to the gauge field and including the $U(1)$ gauge fixing term. Finally, F^2 can be varied with respect to both the metric and the gauge field, giving rise to a mixing term $\mathcal{L}_{ha}$. While performing the variations, the boundary terms cancel the surface terms arising from integration by parts.

Note that we added a star denoting complex conjugation in all expressions. The reason is that it is convenient to allow for complex perturbations $h_{\mu\nu}$ and a_μ. Writing things this way is not strictly necessary but very handy [7], essentially because it is easier to work with exponentials e^{ikx} rather than sines and cosines.

The price we have to pay for fixing the gauge is the appearance of the Faddeev-Popov-determinant in the path integral, which arises because the configuration space splits non-trivially into gauge slices and gauge orbits [25]. This determinant is incorporated into the action by introducing ghost fields. The corresponding Lagrangian can be determined by considering how the gauge fixing conditions $\mathcal{F}$ and $\mathcal{G}$ in (4.5) vary under a gauge transformation with fermionic parameters c, c^μ. The ghost Lagrangian is then [15, 26]

$$\mathcal{L}_{ghost} \propto b^\mu\, \delta\mathcal{G}_\mu + b\,\delta\mathcal{F}\ , \tag{4.8}$$

where $\{b^\mu, c^\mu\}$ and $\{b, c\}$ denote the diffeomorphism and $U(1)$ ghosts, respectively. Using standard expressions for the Lie derivative $\mathcal{L}_X$, we find that to leading order in the fluctuations

$$\delta h_{\mu\nu} = (\mathcal{L}_c \bar{g})_{\mu\nu} = \bar{\nabla}_\mu c_\nu + \bar{\nabla}_\nu c_\mu \; . \tag{4.9}$$

The gauge field picks up a contribution from both the diffeomorphism and the $U(1)$ gauge transformation such that [26]

$$\delta a_\mu = \bar{\nabla}_\mu c + 2(\mathcal{L}_c \bar{A})_\mu = \bar{\nabla}_\mu c - 2\bar{F}_{\mu\nu} c^\nu + 2\bar{\nabla}_\mu (\bar{A}_\nu c^\nu) \; . \tag{4.10}$$

Here, we used Cartan's magic formula $\mathcal{L}_X = \iota_X d + d\iota_X$ in the last step. Thus,

$$\begin{aligned} b^\mu \, \delta\mathcal{G}_\mu + b\, \delta\mathcal{F} &= b^\mu \left(\bar{\Box} c_\mu + \bar{\nabla}_\nu \bar{\nabla}_\mu c^\nu - \bar{\nabla}_\mu \bar{\nabla}_\nu c^\nu \right) + b\bar{\Box} c - 2b \; \bar{\nabla}^\mu (\bar{F}_{\mu\nu} c^\nu) + 2b\bar{\Box}(\bar{A}_\nu c^\nu) \\ &= b^\mu \left(\bar{g}_{\mu\nu} \bar{\Box} + \bar{R}_{\mu\nu} \right) c^\nu + b\bar{\Box} c - 2b \; \bar{F}_{\mu\nu} \bar{\nabla}^\mu c^\nu + 2b\bar{\Box}(\bar{A}_\nu c^\nu) \; . \end{aligned} \tag{4.11}$$

The last term can be removed by shifting $c \to c - 2\bar{A}_\nu c^\nu$. Hence, the ghost Lagrangian (with a convenient normalisation) reads [26, 7]

$$\mathcal{L}_{ghost} = -\frac{1}{16\pi} \left(b_\mu \left(\bar{g}^{\mu\nu} \bar{\Box} + \bar{R}^{\mu\nu} \right) c_\nu + b\bar{\Box} c - 2b \; \bar{F}^{\mu\nu} \bar{\nabla}_\mu c_\nu \right) \; . \tag{4.12}$$

Notice that there is a mixing term that leads to interactions between the $U(1)$ and diffeomorphism ghosts.

Thus, the final expression for the quadratic term in the action is

$$S_E^{(2)} = \int \mathrm{d}^4 x \sqrt{\bar{g}} \left(\mathcal{L}_h + \mathcal{L}_a + \mathcal{L}_{ha} + \mathcal{L}_{ghost} \right) \; . \tag{4.13}$$

It is our goal to compute the determinant of the operator appearing in this expression.

4.3 The conformal mode

The path integral at one-loop order (4.4) is a Gaussian integral that, as long as all eigenvalues λ_i of the operator in (4.13) are positive, can be evaluated directly in terms of the determinant of the operator. This determinant is formally given as a product of eigenvalues

$$\det \hat{O} = \left(\prod \lambda_i \right)_{reg} \tag{4.14}$$

with a suitable regularisation for the (infinite) product [27], to be discussed in section 5. Note that we have to modify (4.4) appropriately to include the ghost contributions. In particular, since the ghosts are fermionic, their determinant contributes to the *numerator* of the final expression.

But what happens if some operator eigenvalues are negative, i.e. if the operator in (4.4) is not positive definite? Then the path integral is no longer Gaussian but badly divergent, since the coefficient in the exponential has the 'wrong' sign (see Fig. 4). As already mentioned in section 3, this is a well-known problem in path integrals involving gravity, in the form of the *conformal mode problem* [17].

How can we deal with this difficulty? The prescription adopted by Gibbons and Hawking [17] is to rotate the integration contour for this conformal mode by hand to the imaginary axis, which introduces an additional minus sign in the exponent that makes the integral convergent. Although this prescription solves the problem of the divergence, the solution is ad hoc and perhaps unsatisfying. One possible resolution is to return to first principles and try to gain a better understanding of the Lorentzian path integral, where

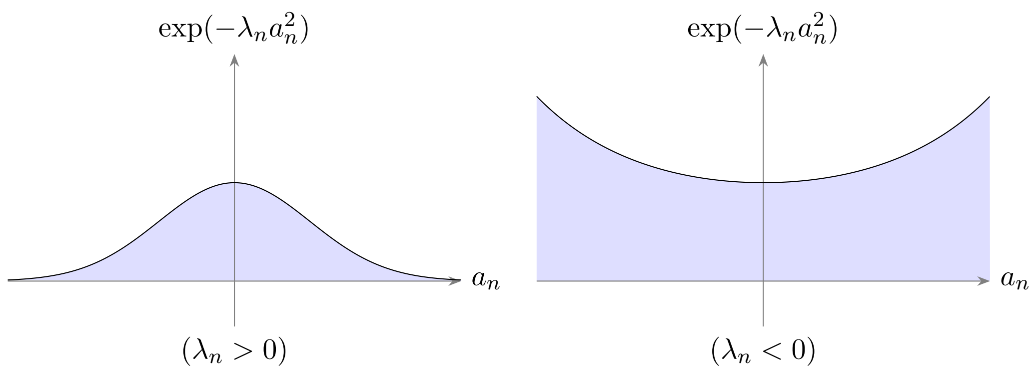

Figure 4: Fluctuations in field space, parametrised by coefficients a_n, with eigenvalue with the 'right' and 'wrong' sign, leading to exponential suppression and enhancement, respectively. A negative eigenvalue introduces a divergence in the (Euclidean) path integral.

the conformal mode is not obviously problematic. It seems plausible that a more complete understanding of Wick rotation and the relationship of the Euclidean and Lorentzian path integral will help us uncover the mysteries of the conformal mode [28].

Having arrived at this point, it seems like we are well on track to compute the one-loop determinant. We reduced the background to an $\mathrm{AdS}_2 \times S^2$ geometry and computed the action to quadratic order. Even though the operator in (4.13) may look daunting, the background is simple and symmetric enough to hope that its eigenvalues can be found, making it possible to compute (4.14).

It is all the more surprising that the universe has once again decided to throw us a curveball. Analysing the differential operator in (4.13), we find that it is much worse behaved than one would naively think. When analysing the sign of the eigenvalues (see Fig. 4), we have secretly swept a third possibility under the rug: The eigenvalue could be exactly zero, which also leads to divergences. Even after including gauge fixing, which we usually expect to remove these 'flat' directions in field space, it will turn out that there is an infinite number of these problematic modes remaining. The reasons for and implications of these zero modes are the subjects of the next sections.

4.4 The role of zero modes

A zero mode is a fluctuation of the fields that is annihilated by the operator (4.13). In other words, it is a flat direction in configuration space (see Fig. 4). It is associated with a mode in the fields that leaves the action invariant, at least to second order in the expansion about the classical solution. We can immediately see why these modes are problematic. A zero eigenvalue in the Gaussian integral means that the integral in this direction is not suppressed exponentially, leading to an infinite contribution.

To gain some intuition for when these zero modes appear, let us start with a simple example: a free scalar in flat spacetime. The Euclidean path integral is given by

$$Z = \mathcal{N} \int \mathcal{D}\phi \, \exp\left(-S_E\right) \,, \quad S_E = \frac{1}{2} \int \mathrm{d}^4 x \, \phi \left(-\Delta + m^2\right) \phi \,, \tag{4.15}$$

where

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4.16}$$

is the (Euclidean) Laplacian in four dimensions. We are thus interested in the eigenvalues of the operator $-\Delta + m^2$. The analysis is particularly simple since a complete basis is known explicitly. A (generalised, δ-function normalised) basis of eigenfunctions for this operator is formed by plane wave solutions [29]

$$\phi_k = \exp(ikx) \; , \quad k \in \mathbb{R}^d \; , \quad \left(-\Delta + m^2\right) \phi_k = \left(k^2 + m^2\right) \phi = \lambda_k \phi \; . \tag{4.17}$$

If $m^2 > 0$, we can see that this eigenvalue is always positive[7]. Thus, there are no zero modes for the massive scalar.

This is different if $m^2 = 0$. Then, the spectrum has a single zero eigenvalue, the constant mode with $\phi_{k=0} = 1$. Note that this mode is not normalisable (not square-integrable), but it is still a generalised eigenfunction that normalises to a δ-function. To anticipate a result discussed in section 4.5, it turns out we do not have to deal with zero modes that normalise to δ-functions in this essay. For our black hole geometry (with our choice of boundary conditions), the zero modes appearing in the path integral live in the *discrete* part of the spectrum which is normalisable in the usual, square-integrable sense [7, 8].

For completeness, let us state explicitly that we require the modes to be sufficiently *regular* to be included in our path integral [7]. This is in line with the discussion in the previous sections. We identified regularity of fields in the path integral as crucial to derive the black hole temperature and entropy (section 3), and we do not give up on it now.

Let us now consider an example where the constant zero mode does appear explicitly in the path integral. An instructive example in this context is scattering in bosonic string theory. The theory is famously governed by the Polyakov action describing a collection of massless scalar fields [30]. Hence, we find one zero mode per scalar field in the path integral.

Computing scattering amplitudes in string theory amounts to evaluating correlation functions for a string worldsheet of different topologies. Consider scattering of m tachyons (the string ground state) at tree level. The relevant topology is a sphere S^2. The amplitude can be written as [30]

$$\mathcal{A}(p_1, ..., p_m) \propto g_s^{m-2} \int \prod_{i=1}^{m} \mathrm{d}^2 z_i \, \langle V(z_1, p_1) ... V(z_m, p_m) \rangle \tag{4.18}$$

with the string coupling g_s. Here, z denotes a complex coordinate on S^2 (thought as $S^2 \cong \mathbb{C} \cup \{\infty\}$ via stereographic projection). The expectation value is computed using the Polyakov action of the bosonic field $X^\mu(z, \bar{z})$ with $\mu \in \{1...26\}$. The external states are described by vertex operators

$$V(z_i, p_i) = \, : \exp(i p_i^\mu X_\mu(z_i, \bar{z}_i)) : \, , \tag{4.19}$$

so that the correlation function reads

$$\langle V(z_1, p_1) ... V(z_m, p_m) \rangle = \int \mathcal{D} X_\mu \exp\left(T \int d^2 z \; X^\mu \partial \bar{\partial} X_\mu + i \sum_{i=1}^{m} p_i^\mu X_\mu(z_i, \bar{z}_i) \right) \tag{4.20}$$

with the string tension T. The operator $\partial\bar{\partial}$, consisting of holomorphic and anti-holomorphic derivatives, is just the two-dimensional Laplacian on the worldsheet.

[7] To achieve a zero eigenvalue, we would need solutions with imaginary k. These exist formally, but grow exponentially and are typically excluded from the operator domain when analysing the spectrum [29].

The appearance of the zero modes means that the operator $\partial\bar{\partial}$ is not invertible, so we cannot solve the Gaussian integral directly. However, because there are only 26 of these problematic modes, it is relatively straightforward to isolate them and treat the zero mode contribution separately. We define the coefficients of the constant modes to be x_0^μ and split the path integral measure as [30]

$$\mathcal{D}X_\mu \to \mathrm{d}^{26}x_0\, \mathcal{D}X'_\mu \ , \tag{4.21}$$

where $\mathcal{D}X'_\mu$ now contains all other (non-zero) modes. The x_0^μ-integration can now be performed explicitly, yielding [30]

$$\mathcal{A}_{zero-mode} = \int \mathrm{d}^{26}x_0 \exp\left(i\sum_{i=0}^{m} p_i \cdot x_0\right) = (2\pi)^{26}\delta\left(\sum_{i=1}^{m} p_i^\mu\right) \ . \tag{4.22}$$

We can see that nothing about the zero modes in bosonic string scattering is fundamentally bad or pathological. Quite the opposite: they are essential to enforce spacetime momentum conservation, an important physical feature of the scattering process.

As a final remark, note that a change in the constant modes x_0^μ corresponds to a translation in spacetime. The zero modes exists because of this physical symmetry. Their effect on the path integral is to enforce conservation of momentum, the Noether charge associated to spacetime translations.

Let us take a step back and summarise the lessons we learned from these examples in quantum field theory and string theory about the appearance and behaviour of zero modes.

1. Many path integrals do not have zero modes: They are highly non-generic. Intuitively, if we take a generic eigenfunction of an operator, it seems quite unlikely that its eigenvalue would be exactly zero.

2. Zero modes are not social animals: if they show up, then usually in small numbers. This is particularly common if the system (or background) has a special feature, symmetry, or behaves non-generically in some way.

3. If the path integral has a small number of zero modes, it is often possible to isolate them and evaluate their contribution separately from the other modes in the path integral. Sometimes, zero mode contributions tell us important things about the underlying physics.

In the next section, we will see if and how we can apply these insights to the extremal black hole.

4.5 Tensor modes of the extremal black hole

Having gained some intuition about zero modes, let us now apply this discussion to the extremal black hole. For a field perturbation to be a zero mode, it must be annihilated by the operator (4.13). We now show that there are an infinite number of zero modes appearing in the extremal path integral, following the construction in [8, 7], which are the main references for this section.

First, note that our current form (2.10) of the AdS$_2 \times S^2$ background metric is not well suited for this calculation. This is because the temperature enters the metric in a slightly awkward way: via the periodicity of the time coordinate. We can cast the background metric into a more convenient form of the by expanding the Euclidean RN metric (3.5) in powers of temperature. The idea is to perform a change of time coordinate [8]

$$\tau = 2\pi T t \ , \tag{4.23}$$

which fixes the periodicity of the time-like coordinate 'by hand' to $\tau \sim \tau + 2\pi$, which is independent of T. We can then perform the extremal limit and the 'zoom-in' to the near-horizon region by sending $T \to 0$. The expansion of the metric in powers of T is presented in detail in Appendix A, following the discussion in [8, 7]. The upshot is that the $\mathsf{AdS}_2 \times S^2$ background takes the form

$$\bar{g} = r_0^2 \, (\underbrace{\sinh^2(\eta)\mathrm{d}\tau^2 + \mathrm{d}\eta^2}_{\mathsf{AdS}_2} + \underbrace{\mathrm{d}\Omega^2}_{S^2}) \; , \tag{4.24}$$

with the coordinates $\eta \in [0, \infty)$ and $\tau \sim \tau + 2\pi$. Notice that we rewrote the AdS_2 part in so-called *de Sitter slicing*, which turns out to be a convenient form. In these coordinates, the background gauge field in the extremal limit is given by

$$\bar{A} = iQ(1 - \cosh\eta)\mathrm{d}\tau \; , \quad \bar{F}_{\tau\eta} = iQ \sinh\eta \; . \tag{4.25}$$

In the following, it will be useful to reference the AdS_2 and S^2 part of the geometry separately by writing

$$\bar{g} = \bar{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b + \bar{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j \; , \quad \bar{A} = \bar{A}_a \mathrm{d}x^a \; , \tag{4.26}$$

where the indices a, b run over AdS_2 and i, j run over the compact S^2.

Let us now determine the zero modes of the system explicitly, first investigating pure metric perturbations that act on AdS_2 and leave the S^2 part alone. These modes are called *tensor modes*. How can we find them explicitly, given the complicated form of the operator (4.13)? First, observe that a metric perturbation h_{ab} on AdS_2 will, by construction, leave the Einstein-Maxwell action invariant if it is a *pure diffeomorphism*

$$h_{ab} = \left(\mathcal{L}_\zeta \bar{g}\right)_{ab} \quad \text{for some } \zeta \; . \tag{4.27}$$

But not all of these fluctuations can be genuine zero-modes. The whole point of our gauge fixing procedure discussed in section 4.2 was to break gauge invariance and remove this redundancy from the path integral. If our modes should stand a chance to be zero modes, they must comply with this choice of gauge. We achieve this by defining the vector field ζ in terms of a scalar field Φ as [7]

$$\zeta^a = \epsilon^{ab}\bar{\nabla}_b\Phi \; . \tag{4.28}$$

Here, ϵ_{ab} is the Levi-Civita tensor on AdS_2. Given this definition, the metric fluctuation is automatically traceless,

$$h = 2\bar{g}^{ab}\bar{\nabla}_{(a}\zeta_{b)} = 2\epsilon_{ab}\bar{\nabla}^a\bar{\nabla}^b\Phi = 0 \; , \tag{4.29}$$

since covariant derivatives commute on scalar fields. To be consistent with the harmonic gauge condition (4.5), the metric perturbation thus has to satisfy

$$\bar{\nabla}^b h_{ab} = 0 \; . \tag{4.30}$$

Using (4.27) and (4.28) as well as the form of the Riemann tensor for (the maximally symmetric space) AdS_2, one can show

$$\bar{\nabla}^b h_{ab} = \epsilon_{ab}\bar{\nabla}^b \left(\Box_{\mathsf{AdS}_2} + \mathcal{R}\right)\Phi \; , \tag{4.31}$$

where $\mathcal{R}$ denotes the Ricci scalar of AdS_2. Thus, we want Φ to solve the equation [7]

$$\left(\Box_{\mathsf{AdS}_2} + \mathcal{R}\right)\Phi = 0 \; . \tag{4.32}$$

So far, everything looks quite normal. This equation is very similar to the Laplace equation for a massive particle (4.17). In our analysis of the flat space case there, we found that there are generally no solutions to this equation that do not blow up at infinity. Does that mean there are no zero modes? The crucial difference to the analysis in section 4.4 is that here we do *not* require the field Φ to satisfy any kind of boundary or normalisability condition. A metric perturbation h_{ab} can satisfy the Dirichlet boundary conditions and hence be perfectly admissible while arising from a function Φ that is badly divergent.

Let us see how this works explicitly. With the background metric (4.24), we find that (4.32) is solved by

$$\Phi_n = e^{in\tau} \tanh^{|n|}(\eta/2) \left(|n| + \cosh\eta\right) , \quad n \in \mathbb{Z} , \tag{4.33}$$

which agrees with the result given in [7] up to a minor misprint. As expected, these scalar functions are clearly divergent and have an infinite norm

$$\int \mathrm{d}\tau \mathrm{d}\eta \sqrt{g}\, |\Phi_n|^2 = \infty . \tag{4.34}$$

Using (4.28), the corresponding vector ζ that generates the diffeomorphism is given by

$$\begin{aligned} \zeta_n = \zeta_n^a \partial_a = \left(\epsilon^{ab} \bar{\nabla}_b \Phi_n\right) \partial_a &= \frac{1}{r_0^2 \sinh\eta} \left(\left(\partial_\eta \Phi_n\right) \partial_\tau - \left(\partial_\tau \Phi_n\right) \partial_\eta \right) \\ &= \frac{1}{r_0^2} e^{in\tau} \tanh^{|n|}(\eta/2) \left(\frac{|n|(|n| + \cosh\eta) + \sinh^2\eta}{\sinh^2\eta} \partial_\tau - in \frac{|n| + \cosh\eta}{\sinh\eta} \partial_\eta \right) . \end{aligned} \tag{4.35}$$

The metric perturbation can now be calculated from (4.27), using the Christoffel symbols of AdS_2. We find

$$\boxed{h_{ab}^{(n)} \mathrm{d}x^a \mathrm{d}x^b = 2in(n^2-1) e^{in\tau} \tanh^{|n|}(\eta/2) \left(\mathrm{d}\tau^2 - \frac{|n|}{n} \frac{2i \mathrm{d}\tau \mathrm{d}\eta}{\sinh\eta} - \frac{\mathrm{d}\eta^2}{\sinh^2\eta} \right) ,} \tag{4.36}$$

which again agrees with the result in [8, 7, 6] up to conventions and minor inconsistencies likely due to typographical errors. We can immediately see that the expression is traceless, as discussed above.

Note that $h_{ab}^{(n)}$ vanishes for $n \in \{0, \pm 1\}$. The vectors ζ_0, ζ_1 and ζ_{-1} hence generate vanishing metric perturbations, i.e. they are the familiar *Killing vectors* (2.12) of AdS_2 [8]. We have

$$\zeta_0 = Q^{-2} \partial_t , \quad \zeta_{\pm 1} = Q^{-2} e^{\pm i\tau} \left(\coth\eta\, \partial_\tau \mp i\, \partial_\eta \right) . \tag{4.37}$$

Their commutators are

$$[\zeta_0, \zeta_{\pm 1}] = \pm i Q^{-2} \zeta_{\pm 1} , \quad [\zeta_{+1}, \zeta_{-1}] = -2i Q^{-2} \zeta_0 , \tag{4.38}$$

so after rescaling the vectors we recover the familiar $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra structure.

Are these modes compatible with the *boundary conditions* we set up for the path integral? First, notice that the components of the fluctuation h_{ab} are suppressed compared to the background metric $\bar{g}$. The $\eta\eta$-component and the off-diagonal component decay exponentially fast as $\eta \to \infty$, and the $\tau\tau$-component is of $\mathcal{O}(1)$ while $\bar{g}_{\tau\tau} \sim \sinh^2\eta$. Thus, the fluctuations *respect the Dirichlet boundary condition* of the path integral. We can also calculate their norm via the inner product defined as

$$\langle h^{(m)} | h^{(n)} \rangle \equiv \int \mathrm{d}^4x \sqrt{\bar{g}}\, h_{\mu\nu}^{(m)*} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}^{(n)} . \tag{4.39}$$

Using (4.36), this evaluates to

$$\begin{aligned} \langle h^{(m)} | h^{(n)} \rangle &= \int \mathrm{d}^4x \sqrt{\bar{g}}\, h_{\mu\nu}^{(m)*} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}^{(n)} \\ &= 128\pi^2 n^2 (n^2-1)^2 \delta_{mn} \int_0^\infty \mathrm{d}\eta\, \tanh^{2|n|}(\eta/2) \sinh^{-3}\eta \\ &= 32\pi^2 |n| (n^2-1) \delta_{mn} \quad \text{if } |n| \geq 2 . \end{aligned} \tag{4.40}$$

The integral can be calculated using section 3.512 of [31]. Thus, as already mentioned in section 4.4, we find that our zero modes, which are part of the discrete part of the spectrum, have finite norm [7].

What about the Neumann boundary conditions for the gauge field? For the metric perturbations to be admissible, they should *preserve the charge Q of the configuration*. In other words, the value of $F_{\mu\nu}$ at infinity should not change. We can verify this explicitly using (4.35), (4.25), and Cartan's magic formula.

$$\begin{aligned}\delta F_{\tau\eta} &= (\mathcal{L}_\zeta \bar{F})_{\tau\eta} = \mathrm{d}(\iota_\zeta \bar{F})_{\tau\eta} = \partial_\tau(\zeta^\tau \bar{F}_{\tau\eta}) + \partial_\eta(\zeta^\eta \bar{F}_{\tau\eta}) \\ &\propto n e^{in\tau} \bar{F}_{\tau\eta} \tanh^{|n|}(\eta/2) \frac{|n|(|n| + \cosh\eta) + \sinh^2\eta}{\sinh^2\eta} \\ &\quad - e^{in\tau}\partial_\eta \left(iQ \tanh^{|n|}(\eta/2) n(|n| + \cosh\eta) \right) = 0 \,,\end{aligned} \tag{4.41}$$

so the variation of the field strength tensor[8] vanishes identically. Thus, the metric perturbation is compatible with the boundary conditions.

To summarise, we have constructed an infinite set of (regular) metric fluctuations (4.36), following the discussion in [7, 8]. These modes are generated by a diffeomorphism ζ, hence they leave the Einstein-Maxwell action invariant. The modes comply with the choice of gauge, so they leave the gauge fixing term invariant. Furthermore, they are compatible with the boundary conditions. Hence, we found an *infinite set of zero modes* of the path integral of the extremal black hole!

As a non-trivial consistency check, we can verify that (4.36) are zero modes explicitly using the operator in (4.7). The expression for $D^{\mu\nu\rho\sigma}$ simplifies considerably because of the direct product structure of the background, and because the modes h_{ab} only have support on AdS$_2$. In particular, the curvature tensors and differential operators cleanly separate into two parts AdS$_2$ and S^2. Hence, the S^2 part effectively plays no role in the computation. Using (4.27), (4.28) and the curvature tensors of AdS$_2$, one can show identities like

$$\bar{\Box} h_{ab}^{(n)} = \mathcal{R} h_{ab}^{(n)} = -2Q^{-2} h_{ab}^{(n)} \tag{4.42}$$

to evaluate the derivative terms in $D^{\mu\nu\rho\sigma}$. The terms involving $\bar{F}_{ab}$ can be evaluated using (4.25). Combining all contributions, we find

$$D^{\mu\nu ab} h_{ab}^{(n)} = 0 \,. \tag{4.43}$$

Thus, D annihilates the modes in (4.36), confirming our earlier claim based on symmetry arguments.

The appearance of tensor modes is very surprising. According to the conclusions in section 4.4, we would expect a generic path integral to have no zero modes, or maybe a small finite number in special cases or geometries. For the extremal black hole, we are dealing with an infinite number of them.

How can we interpret this result? Have we made a mistake? One might suspect that the appearance of zero modes is a result of some remaining gauge freedom that was not properly fixed (cf. section 4.2). Could we eliminate the zero modes if we added a better gauge fixing term to the action? To answer this question, we can investigate the vector field ζ that generates our diffeomorphism a bit closer. Importantly, at $\eta \to \infty$, the vector field does not fall off,

$$\lim_{\eta\to\infty} \zeta_n \propto \partial_\tau - in\partial_\eta \,. \tag{4.44}$$

[8] Note that F is proportional to the volume form of AdS$_2$. Hence, we have shown that the ζ_n generate volume preserving diffeomorphisms.

Like the scalar field Φ, the vector ζ is non-normalisable, and the diffeomorphism acts non-trivially at infinity. These transformations are sometimes called 'large diffeomorphisms'. They carry non-trivial physical information and cannot simply be eliminated by gauge fixing (cf. [8]).

The upshot of this discussion is the following. The $\mathrm{AdS}_2 \times S^2$ background is special because it admits *large diffeomorphisms* that yield *decaying metric perturbations*. This structure is a key part of the mechanism that gives rise to the infinite tower of zero modes.

Recall that in this section we restricted ourselves to tensor modes that act purely on the AdS_2 part, with the S^2 unaffected. It turns out that there are other infinite families of zero modes that rely on a similar mechanism as the tensor modes. These are *vector modes* h_{ai}, which are a combination of vectors on AdS_2 and a Killing vector on S^2, and *gauge modes* a_μ, which are perturbations of the gauge field. These modes are discussed in detail in [7]. Furthermore, one might suspect that the ghost fields give rise to additional zero modes. It was shown in [7] that this is not the case.

We have come a long way in describing the one-loop quantum corrections to the path integral of an extremal black hole. After setting up the problem and discussing general properties of the path integral, we computed the relevant operator appearing in the second order expansion of the action. But it turns out that extremal black holes are more subtle than one might think, and their path integral includes an infinite number of problematic zero modes. This means that the usual methods of dealing with zero modes fail (cf. section 4.4), so we do not have a good handle on the path integral in its current form.

How can we deal with this unexpected obstacle? The next-best thing to try is to take the black hole slightly away from extremality by raising the temperature to a small but non-zero value [7, 8]. This means that the near-horizon geometry is no longer exactly $\mathrm{AdS}_2 \times S^2$, and the eigenvalue of the zero modes pick up a temperature-dependent correction, allowing us to compute the (Gaussian) path integral. We can then analyse the system at very small temperature. This calculation is the subject of the next section.

5 The Near-Extremal Path Integral

Having discovered that the extremal path integral contains an infinite number of zero modes, we follow [6, 7] and move away slightly from the extremal limit, to the so-called near-extremal black hole. This breaking of the $\mathrm{AdS}_2 \times S^2$ symmetry induces a change $\delta\Lambda$ in the eigenvalues of the zero modes, effectively regularising the eigenvalues. If $\delta\Lambda > 0$, this restores the Gaussian path integral and allow us to compute it. We can then examine how the partition function of the black hole behaves in the low-temperature regime.

5.1 Perturbed eigenvalues in the near-extremal background

The near-extremal metric is obtained by taking the small-temperature limit of Euclidean RN, where we now keep terms up to linear order in the temperature, and restricting to the near-horizon region. It can be understood as a small perturbation to the $\mathrm{AdS}_2 \times S^2$ geometry. The metric is (A.7)

$$\begin{aligned}\bar{g} = {} & r_0^2\,(\underbrace{\sinh^2(\eta)\mathrm{d}\tau^2 + \mathrm{d}\eta^2}_{\mathrm{AdS}_2} + \underbrace{\mathrm{d}\Omega^2}_{S^2}) \\ & + 4\pi Q^3 T\left((2+\cosh\eta)\tanh^2(\eta/2)(\mathrm{d}\eta^2 - \sinh^2\eta\,\mathrm{d}\tau^2) + \cosh\eta\,\mathrm{d}\Omega^2\right)\end{aligned} \tag{5.1}$$

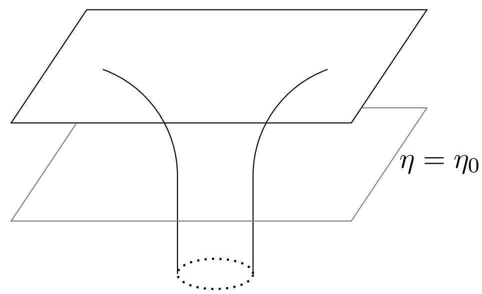

Figure 5: The geometry of the near-extremal black hole. The throat has finite length, and the metric (5.1) is valid up to some large value η_0.

with the gauge field (A.8)

$$\begin{aligned}\bar{A} &= iQ(1 - \cosh\eta + 2\pi QT \sinh^2\eta)\,\mathrm{d}\tau \\ \bar{F}_{\tau\eta} &= iQ\sinh\eta - 4\pi i Q^2 T \sinh\eta\cosh\eta \;.\end{aligned} \tag{5.2}$$

Physically, this perturbation means that the black hole throat, which previously was infinitely long (see Fig. 1), now has finite length. By zooming in to the near-horizon region, we have implicitly cut off this throat at a large yet finite radius η_0 (see Fig.5). The expressions (5.1) and (5.2) are valid in the regime $e^\eta \ll (QT)^{-1}$ (see Appendix A).

It is a priori not clear that the near-horizon region is still a good approximation for the physical behaviour of the black hole, although the fact that we are only considering a small deviation from AdS$_2 \times S^2$ with a throat that is still large is reason enough to be optimistic. There has also been some progress in deriving the results presented in this section independently of the near-horizon geometry [32].

The correction $\delta\Lambda_{tensor}$ to the tensor mode eigenvalues can be calculated using standard arguments of quantum mechanical perturbation theory [7]. To linear order in T, both the operator D in (4.7) and the zero modes $h^{(n)}$ pick up a correction

$$\begin{aligned} D &\to \bar{D} + \delta D \;, \\ h^{(n)} &\to h^{(n)} + \delta h^{(n)} \;. \end{aligned} \tag{5.3}$$

They are related to the corrected eigenvalue by

$$(\bar{D} + \delta D)(h^{(n)} + \delta h^{(n)}) = (\Lambda_n + \delta\Lambda_n)(h^{(n)} + \delta h^{(n)}) \;, \tag{5.4}$$

which at linear order reads

$$\bar{D}\delta h^{(n)} + \delta D h^{(n)} = \Lambda_n \delta h^{(n)} + \delta\Lambda_n h^{(n)} \;. \tag{5.5}$$

Taking the inner product of this expression with $\langle h^{(n)}|$ and using that D is Hermitian gives

$$\delta\Lambda_n = \frac{\langle h^{(n)}|\delta D|h^{(n)}\rangle}{\langle h^{(n)}|h^{(n)}\rangle} = \frac{1}{32\pi^2|n|(n^2-1)} \int \mathrm{d}^4x\sqrt{\bar{g}}\, h^{(n)*}_{\mu\nu}\delta D^{\mu\nu\rho\sigma} h^{(n)}_{\rho\sigma} \;, \tag{5.6}$$

which is nothing but the familiar statement from quantum mechanics that the first order correction to an eigenvalue is the change in the operator evaluated on the unperturbed eigenstate.

According to textbook quantum mechanics, the expression (5.6) for the perturbed eigenvalue usually only holds in non-degenerate perturbation theory [33]. If the unperturbed eigenstates are degenerate, which is our case of interest, the eigenstates of the full operator D does generally not approach a single mode $h^{(n)}$ but a linear combination of them as $T \to 0$, invalidating our perturbative expansion (5.3) [33]. In the case of the tensor modes, the fact that $h^{(n)} \propto e^{in\tau}$ ensures that the $h^{(n)}$-basis of the degenerate subspace diagonalises δD, which is why the simple formula (5.6) still applies.

Equation (5.6) can be evaluated explicitly by varying D in (4.7) with respect to the change in the background (5.1) and (5.2). For the tensor modes, we find

$$(\delta\Lambda_{tensor})_n = \frac{|n|T}{16Q} + \mathcal{O}(T^2) \, . \tag{5.7}$$

The computation was performed using symbolic computation in Mathematica (using the diffgeo package), and the result agrees with the findings in [7].

If we include the other families of zero modes in the picture (vector and gauge modes), this result still holds [7], but there is an additional contribution from the vector modes. The gauge modes behave differently, as their eigenvalue is not corrected at linear order in T [6].

As we can see, the correction to the eigenvalue is indeed positive, which means that (5.7) can be used to perform the Gaussian integral. The tensor mode contribution to the logarithm of the partition function is (using (4.4) and (4.14))

$$\delta \log Z_{\mathsf{tensor}} = -\frac{1}{2}\log\det \hat{O}_{\mathsf{tensor}} = -\frac{1}{2}\operatorname{Tr}\log \hat{O}_{\mathsf{tensor}} = -\frac{1}{2}\sum_n \log(\delta\Lambda_n) \, . \tag{5.8}$$

Naturally, this correction is divergent in the extremal limit $T \to 0$. In fact, the contribution of the zero modes yields the *dominant temperature-dependent contribution* to the partition function. This is because the contributions to the one-loop determinant from the non-zero modes is smooth/non-singular in the limit $T \to 0$. If we are interested in the temperature dependence of the partition function in the near-extremal case, it is thus sufficient to consider only the zero mode contributions [7].

5.2 Regularisation and one-loop determinant

We are now left with the expression

$$\delta \log Z_{\mathsf{tensor}} = -\frac{1}{2}\sum_n \log(\delta\Lambda_n) = -\sum_{n\geq 2} \log an \, , \quad a = \frac{T}{16r_0} \tag{5.9}$$

for the one-loop contribution to the partition function coming from the tensor modes. The sum is formally divergent because of the infinite number of modes involved. This is very common in the context of quantum field theory [34], and it is possible to *regularise* this sum by systematically removing divergent pieces and assigning it a finite value. One common and effective way to achieve this is zeta function regularisation, which was introduced in the quantum field theory context by Hawking [27]. The idea of this prescription is to cast the sum into the form of the series expansion of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \, . \tag{5.10}$$

This form of the sum is still divergent for the physically relevant value of s. However, the zeta function has a well known analytic continuation for all s, allowing us to essentially define the original sum by this analytic continuation.

Let us see how this works in practice. We write

$$\begin{aligned}\delta \log Z_{\text{tensor}} &= -\sum_{n\geq 2} \log an = \log a - \sum_{n=1}^{\infty} \log an = \log a + \frac{\mathrm{d}}{\mathrm{d}s} \sum_{n=1}^{\infty} \frac{1}{(an)^s}\Bigg|_{s=0} \\ &= \log a + \frac{\mathrm{d}}{\mathrm{d}s}\left(a^{-s}\zeta(s)\right)\Bigg|_{s=0} = (1-\zeta(0))\log a + \zeta'(0)\ .\end{aligned} \tag{5.11}$$

We can now use the values of the zeta function and its derivative, analytically continued to zero. This gives [7]

$$\delta \log Z_{\text{tensor}} = \frac{3}{2}\log a - \frac{1}{2}\log 2\pi = \frac{3}{2}\log\frac{T}{r_0} - \log\left(64\sqrt{2\pi}\right)\ . \tag{5.12}$$

This way of turning an infinite sum into a finite result may seem unconvincing and slightly ad hoc. However, a key aspect of regularisation is that the same finite piece can be extracted from an infinite sum independently of the procedure chosen [35]. To demonstrate this, let us re-evaluate the sum (5.12) using a different regularisation prescription. Instead of zeta function regularisation, we introduce a regulator function $\exp(-\epsilon n)$ in the sum and extract the result as the constant term in the series expansion of the result in ϵ. The temperature-dependent part of the infinite series in (5.9) reads

$$-\log a \sum_{n=1}^{\infty} e^{-\epsilon n} = -\log a\left(1 - \sum_{n=0}^{\infty}\left(e^{-\epsilon}\right)^n\right) = -\frac{\log a}{e^{\epsilon}-1} = -\log a\left(\frac{1}{\epsilon} - \frac{1}{2} + \mathcal{O}(\epsilon)\right)\ , \tag{5.13}$$

after evaluating the geometric series. We now define the series by its finite piece, i.e.

$$-\log a \sum_{n=1}^{\infty} 1 \to \frac{1}{2}\log a\ . \tag{5.14}$$

Thus, we confirmed that we can recover the result

$$\delta \log Z_{\text{tensor}} = \frac{3}{2}\log a + \{T\text{-independent}\} \tag{5.15}$$

by using a very different regularisation procedure.

As a final comment, let us mention that the idea of regularising sums is well-established in various quantum field theory contexts. For example, it can be used to compute the Casimir force due to vacuum fluctuations between two plates [35], an effect that has been confirmed experimentally.

One can analyse the contribution from the other families of zero modes using methods similar to those presented here for the tensor modes. The upshot is that the vector modes give the same $3/2\,\log T$ contribution as the tensor modes, while the gauge modes do not contribute a divergent piece to $\log Z$ [6]. Thus, the final result for the one-loop correction to the partition function is

$$\delta \log Z_{\text{1-loop}} = \delta \log Z_{\text{tensor}} + \delta \log Z_{\text{vector}} + \{\text{regular}\} = 3\log\frac{T}{r_0} + \{\text{regular}\}\ . \tag{5.16}$$

Using (3.22) and reinstating fundamental constants, the change in the black hole entropy at one-loop level is [6, 7]

$$\boxed{\delta S_{BH} = 3k_B \log \frac{k_B T G}{c^4 r_0} + \{\text{regular}\} \ .} \tag{5.17}$$

This is a striking result. In quantum field theory, we would usually expect the lower-order, classical contribution to a physical quantity to dominate, with the higher-order quantum effects giving small corrections to this result. This is in agreement with the structure of the formula (4.4), where the classical action contributes exponentially while the one-loop determinant appears only through powers. However, we see that the opposite is true for the black hole near extremality: at low temperature, the *quantum corrections dominate*, as the one-loop term (5.17) completely changes the qualitative behaviour of the entropy.

This results answers the question raised in section 3.3 about the classically large entropy $S = A/4$ near the apparent black hole ground state at $T = 0$ [19]. Firstly, due to the quantum corrections, the entropy is of order one at temperature $T \sim Q \exp(-A/12)$. This means that, contrary to the result in section 3.3, the gravitational path integral does *not* predict a large entropy for near-extremal black holes. Secondly, in the limit $T \to 0$, the entropy diverges, signalling a complete breakdown of the semiclassical description at $T = 0$. With our current tools, we are therefore not able to make reliable predictions about the nature of the ground state of the black hole. Therefore, we conclude that extremal black holes as the ground state of the RN solution do not exist in the way the classical intuition would suggest [19].

6 Conclusions and Discussion

The richness of black holes

We are now at the end of our tour through the world of Reissner-Nordström black holes from the perspective of the gravitational path integral. What have we learned?

Classically, black holes appear to be objects that are incredibly constrained. In four dimensions with spherical symmetry, they can be fully characterised by just two numbers, their charge and their mass. However, once quantum physics enters the picture, many new phenomena arise. Black holes are not actually black, they radiate, and their temperature [1]

$$T = \frac{\hbar \kappa}{2\pi c k_B} \tag{6.1}$$

is fixed purely by regularity conditions on the Euclidean path integral. Moreover, following the method of Gibbons and Hawking, we can derive their entropy by evaluating the Euclidean path integral at tree level [5] (see section 3)

$$Z = \mathcal{N} \int \mathcal{D}g_{\mu\nu} \mathcal{D}A_\mu \, \exp(-S_E[g, A]) \approx \exp(-S_E[\bar{g}, \bar{A}]) \ . \tag{6.2}$$

This yields the famous Bekenstein-Hawking result [3]

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar} \ . \tag{6.3}$$

Thus, black holes are thermodynamic systems, and we find that they obey analogues of the laws of thermodynamics [12].

More surprises appear at loop-level. When computing the one-loop determinant, we find that the path integral at zero temperature contains an *infinite number of zero modes*. We can nevertheless analyse the path integral of the *near-extremal black hole*, where the zero-modes pick up a small temperature-dependent correction. We find a correction to the near-extremal black hole entropy [6, 7]

$$\delta S_{BH} = 3k_B \log \frac{k_B T G}{c^4 r_0} + \{\text{regular}\} \ . \tag{6.4}$$

This means that quantum effects are dominant for very low temperature, which subverts our classical expectations.

It seems like, with black holes, you never really know what you get. So prepare for a surprise!

A nod to universality

One might wonder whether these results, especially the $\log T$ contribution to the entropy, are highly specific to the four-dimensional Reissner-Nordström solution. This would certainly cast some doubt on the physicality and relevance of this result.

Interestingly, similar behaviour has been found in other black hole geometries, for example in BTZ black holes [32] or charged black holes in de Sitter space [7]. In both cases, the path integral (in certain limits) contains infinitely many zero modes that lead to a $\log T$ contribution to the entropy. Therefore, our result seems to be reasonably *universal*.

We can also understand this point from the language of symmetries. The appearance of zero modes can be traced back to the special throat geometry and the *symmetry enhancement*

$$\mathbb{R} \times SO(3) \to SL(2,\mathbb{R}) \times SO(3) \tag{6.5}$$

in the near-horizon region. An enhancement like this is a very common feature among black holes (cf. [7, 19]), which is why we would expect many results obtained from Reissner-Nordström to carry over to other black hole solutions.

There are reasons to be cautious, however. After all, a full understanding of the gravitational path integral for the RN black hole is out of reach. With the one-loop corrections dominating over the tree-level contribution, there is no obvious reason why corrections at higher loop-order should not change the qualitative behaviour of the black hole again. Our physical understanding of low-temperature black holes is far from complete, and there is undoubtedly a lot left for us to uncover in future research.

The future of the gravitational path integral

Progress on the topic of low-temperature black holes will likely go hand in hand with a better understanding of the gravitational path integral as a whole. And while the Euclidean path integral certainly remains the main tool for practical calculations, it is also worthwhile to investigate the Lorentzian path integral directly.

The problems with the Euclidean path integral are easily stated. The Wick rotation to Euclidean time is hard to define unambiguously in curved spacetime [16], and it seems to violate the spirit of general relativity. Furthermore, the Euclidean path integral comes with problems like the conformal mode [17]. Even if these technical difficulties are dealt with, the Euclidean path integral obscures the causal structure of the theory and is hard to justify from first principles (cf. [16]).

That said, the highly oscillatory nature of the Lorentzian path integral means that it is very hard to make sense of in practice, which is arguably a worse problem. However, there have been attempts in recent years to get a handle on Lorentzian path integrals directly. Progress has been made defining the Lorentzian path integral systematically using tools from complex analysis, most notably Picard-Lefschetz theory. This has been applied successfully to path integrals in cosmology [36]. While this method is so far mostly applicable to integrals with very few degrees of freedom, it could be fruitful to investigate this line of research further in the context of the low-temperature black hole path integral.

We can look forward to new results and discoveries in the next decades. We live in an exciting time for theoretical physics, and it is safe to say that the last word on near-extremal black holes has not yet been spoken.

A The Near-Horizon Geometry

In this appendix, we derive the extremal and near-extremal background used in sections 4 and 5 in detail. Our strategy is to expand the Euclidean RN metric (3.5) in powers of temperature T, following the discussion in [8, 7].

The mass and charge of the RN black hole satisfy the relations [6]

$$M = \frac{1}{2r_+}(Q^2 + r_+^2) \ , \quad Q^2 = r_+ r_- \ , \tag{A.1}$$

while its temperature is given by (3.9)

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{1}{4\pi r_+^3}(r_+^2 - Q^2) \ . \tag{A.2}$$

Since we work in the canonical ensemble, the charge is fixed to $Q = r_0$, the radius of the extremal black hole. The dependence of the radius r_+ on the temperature T can be computed by inverting (A.2). Writing r_+ as a power series in T, plugging this ansatz into (A.2) and comparing coefficients, we find

$$r_+ = Q + 2\pi Q^2 T + 10\pi^2 Q^3 T^2 + 64\pi^3 Q^4 T^3 + \mathcal{O}(T^4) \ . \tag{A.3}$$

Using (A.1), we can expand the mass as

$$M = Q + 2\pi^2 Q^3 T^2 + 16\pi^3 Q^4 T^3 + \mathcal{O}(T^4) \ . \tag{A.4}$$

Our result for r_+ agrees with the expansion reported in [6] but differs slightly from the result in [7]. The expansion of M agrees with the result in [7, 6], but not [8]. The reasons for the minor discrepancies are likely typographical errors.

As a quick consistency check, we can use the variations (A.3) and (A.4) to verify that the first law of black hole thermodynamics (3.24) is indeed satisfied.

We now perform the substitution [8]

$$\tau = 2\pi T\, t \ , \quad r = r_+(T) + 2\pi Q^2 T(\cosh\eta - 1) \ . \tag{A.5}$$

With these new coordinates (τ, η) with $\eta \in [0, \infty)$, the periodicity of the time-like coordinate is fixed 'by hand' to $\tau \sim \tau + 2\pi$, which is independent of T.

Furthermore, T now appears explicitly in the metric, making it easy to take the low-temperature limit. Because of $\mathrm{d}\tau = 2\pi T \, \mathrm{d}t$ and $\mathrm{d}r = 2\pi Q^2 T \sinh\eta \, \mathrm{d}\eta$, we must expand the function $f(r)$ to order T^3 to determine the correction linear in T. We find that

$$f(r) = 4\pi^2 Q^2 T^2 \sinh^2\eta \left(1 - 4\pi Q T (2 + \cosh\eta) \tanh^2(\eta/2)\right) + \mathcal{O}(T^4) \ . \tag{A.6}$$

Hence, the background metric in the near-horizon region to linear order in T is [7, 8]

$$\begin{aligned} \bar{g} &= r_0^2 \, (\underbrace{\sinh^2(\eta)\mathrm{d}\tau^2 + \mathrm{d}\eta^2}_{\mathsf{AdS}_2} + \underbrace{\mathrm{d}\Omega^2}_{S^2}) \\ &\quad + 4\pi Q^3 T \left((2 + \cosh\eta) \tanh^2(\eta/2)(\mathrm{d}\eta^2 - \sinh^2\eta \, \mathrm{d}\tau^2) + \cosh\eta \, \mathrm{d}\Omega^2\right) \ . \end{aligned} \tag{A.7}$$

Similarly, we can find an expression for the gauge field. To first order in T, we find

$$\begin{aligned} \bar{A} &= iQ(1 - \cosh\eta + 2\pi Q T \sinh^2\eta) \, \mathrm{d}\tau \\ \bar{F}_{\tau\eta} &= iQ \sinh\eta - 4\pi i Q^2 T \sinh\eta \cosh\eta \ , \end{aligned} \tag{A.8}$$

which agrees with [7, 8, 6] up to conventions. This geometry is valid in the near-horizon region, which translates to the condition $e^\eta \ll (QT)^{-1}$.

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